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The set of equivalent classes of invariant star products on $(G;\beta_1)$

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Abstract

This article, in conjunction with a previous one, proves Drinfeld's theorems about invariant star products, ISPS, on a connected Lie group $\boldsymbol{G}$ endowed with an invariant symplectic structure $\beta_1 \in \mathcal{Z}^2(\mathfrak{g})$. In particular, we prove that every formal 2-cocycle $\beta_\hbar \in \beta_1 + \hbar \cdot \mathcal{Z}^2(\mathfrak{g}))[[\hbar]]$ determines an ISP, $F^{\beta_\hbar}$, and conversely any ISP, F, determines a formal 2-cocycle $\omega_\hbar \in \beta_1 + \hbar \cdot \mathcal{Z}^2(\mathfrak{g})[[\hbar]]$ such that F is equivalent to $F^{\omega_\hbar}$. We also prove that two ISPS $F^{\beta_\hbar}$ and $F^{\omega_\hbar}$ are equivalent if and only if the cohomology classes of $\beta_\hbar$ and $\omega_\hbar$ coincide. These properties define a bijection between the set of equivalent classes of ISP on $(\boldsymbol{G};\beta_1)$ and the set $\beta_1 + \hbar \cdot \mathcal{H}^2(\mathfrak{g})[[\hbar]]$.

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Dedicated to André Lichnerowicz with admiration, gratitude and affection

1. Introduction

1.1

Let $\boldsymbol{G}$ be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathcal{H}^2(\mathfrak{g})$ be the second cohomology space of $\mathfrak{g}$ with respect to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$. Let $\beta_1 \in \mathcal{Z}^2(\mathfrak{g})$ be a cocycle in the above cohomology such that mapping $\tilde{\beta}_1 : \mathfrak{g} \to \mathfrak{g}^*$, where $\tilde{\beta}_1(x) \cdot y = \beta_1(x;y)$,

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$x, y \in \mathfrak{g}$, is an isomorphism. We write $(G; \beta_1)$ for the Lie group G endowed with the left-invariant symplectic structure defined by β_1.

1.2

The aim of this paper is to prove a theorem stated by Drinfeld [11] whose main parts may be defined as follows:

(a) Any formal 2-cocycle $\beta_\hbar \in \beta_1 + \hbar \cdot \mathcal{Z}^2(\mathfrak{g})[[\hbar]]$ determines, on $(G; \beta_1)$, an ISP $F^{\beta_\hbar}(x; y) \in \mathfrak{A}(\mathfrak{g})^{\otimes 2}[[\hbar]]$.

(b) Any ISP $F'(x; y)$ determines a formal 2-cocycle $\beta'_\hbar \in \beta_1 + \hbar \cdot \mathcal{Z}^2(\mathfrak{g})[[\hbar]]$ and is equivalent to the ISP $F^{\beta'_\hbar}(x; y)$ determined by this cocycle.

(c) The ISPS $F^{\beta_\hbar}$, $F^{\omega_\hbar}$ determined, respectively, by cohomologous cocycles $\beta_\hbar$ and $\omega_\hbar = \beta_\hbar + \tilde{\delta}\alpha_\hbar$, $\alpha_\hbar \in \hbar \cdot \mathfrak{g}^*[[\hbar]]$, are equivalent.

(d) If star products $F^{\beta_\hbar}$ and $F^{\beta'_\hbar}$, $\beta'_\hbar \in \beta_1 + \hbar \cdot \mathcal{Z}^2(\mathfrak{g})[[\hbar]]$, are equivalent there exists $\alpha_\hbar \in \hbar \cdot \mathfrak{g}^*[[\hbar]]$ such that $\beta'_\hbar = \beta_\hbar + \tilde{\delta}\alpha_\hbar$.

The above properties define a bijection between the set of equivalent classes of ISPS on $(G; \beta_1)$ and the set $\beta_1 + \hbar \cdot \mathcal{H}^2(\mathfrak{g})[[\hbar]]$.

We have proved this theorem in [24], where we gave explicit proofs for parts (a)–(c), but not for (d).

From these invariant star products we can get [11,22,29] the corresponding triangular quantum groups.

1.3

In Section 5 of this paper, we provide the proof for part (d). To do this, we need, in particular, more to look at the equivalence in part (c), closely discussed in Section 4. In Section 3, we recall the main idea to be developed for the proof of the theorem, briefly describe the proof of parts (a) and (b) and state some intermediary results. In Section 2, we give some necessary background.

The following theorem is clear from (a)–(d):

Theorem 1 (Drinfeld [11]). *Choose a vector subspace V in $\mathcal{Z}^2(\mathfrak{g})$, the space of invariant de Rham 2-cocycles on $\mathfrak{g}$, which is a supplementary space of de Rham 2-exact cocycles $\mathcal{B}^2(\mathfrak{g})$, i.e. $\mathcal{Z}^2(\mathfrak{g}) = V \oplus \mathcal{B}^2(\mathfrak{g})$. Let $F'(x; y)$ be any invariant star product on $(G; \beta_1)$. Then, $F'(x; y)$ is equivalent to one obtained in* (a) *from a cocycle*

$$\beta_\hbar = \beta_1 + \beta_2 \hbar + \cdots + \beta_R \hbar^{R-1} + \cdots,$$

such that $\beta_k \in V$ if $k > 1$. Moreover, $\{\beta_k\}$ is uniquely determined by $F'(x; y)$.

Clearly, we can identify the set $\beta_1 + \hbar \cdot V[[\hbar]]$ with the set $\beta_1 + \hbar \cdot \mathcal{H}^2(\mathfrak{g})[[\hbar]]$ through the bijection

$$[\beta] \in \mathcal{H}^2(\mathfrak{g}) \to v = \beta - \tilde{\delta}\alpha \in V.$$

1.4

In the case of a general symplectic manifold $(\boldsymbol{M}; \omega_1)$, two star products that are equivalent to the order m are equivalent to the order $m+1$ if and only if one specific Hochschild 2-cocycle k_{m+1} is a coboundary [1,17]. The theorem is also true in the case of the set of closed star products and closed equivalence [6], where the theory is now controlled by the cyclic cohomology defined by the condition of closedness.

The point of the proof of (b) is that if F and $F^{\beta_\hbar}$, where $\beta_\hbar = \beta_1 + \beta_2 \hbar + \cdots + \beta_m \hbar^{m-1}$, are equivalent to the order m then k_{m+1} is also an invariant Poisson 2-cocycle and determines [20,24] an invariant cocycle β_{m+1}. If cocycle $\beta'_\hbar = \beta_\hbar + \beta_{m+1} \hbar^m$ is defined, it can be proved that F and $F^{\beta'_\hbar}$ are equivalent to the order $m+1$. This is how the obstruction to the equivalence can be removed at every order.

1.5

De Wilde and Lecomte [8,9] and Fedosov [12,13] have proved that on any symplectic manifold $(\boldsymbol{M}; \omega_1)$ there exists a star product. We refer to [2,10,12,14,25,26] for the proof of the theorem stating that the set $\omega_1 + \hbar \cdot \mathcal{H}^2(\boldsymbol{M})[[\hbar]]$ classifies the equivalence classes of the star products on the manifold $(\boldsymbol{M}; \omega_1)$.

1.6

For the classification of the equivalence classes of 1-differential *infinitesimal* deformations of the *dynamical* Lie algebra $(C^\infty(\boldsymbol{M}); \{\,;\,\}_1)$, the reader is referred to [15]. A bijection has been constructed between the equivalence classes of these deformations and the set

$$\mathcal{P}^1(\boldsymbol{M}; \omega_1) \oplus \mathcal{H}^2(\boldsymbol{M})/\mathcal{Q}^2(\boldsymbol{M}; \omega_1),$$

where $\mathcal{P}^1(\boldsymbol{M}; \omega_1) = \operatorname{Im}(\omega_1 \wedge : \mathcal{H}^1(\boldsymbol{M}) \to \mathcal{H}^3(\boldsymbol{M}))$ and $\mathcal{Q}^2(\boldsymbol{M}; \omega_1) = \operatorname{Ker}(\omega_1 \wedge : \mathcal{H}^2(\boldsymbol{M}) \to \mathcal{H}^4(\boldsymbol{M}))$. As with the classification of ISPS, it can be introduced the symplectic structures defined by the closed 2-forms on $\boldsymbol{M}$, $\omega_\hbar = \omega_1 + \hbar \cdot \omega_2$. The deformation $(C^\infty(\boldsymbol{M})[[\hbar]]; \{\,;\,\}_\hbar)$, may then be defined. A bijection can be constructed between the set of *infinitesimal* deformations which *are not* equivalent to one defined by some $\omega_\hbar$, and the set $\mathcal{P}^1(\boldsymbol{M}; \omega_1)$. It is also possible to prove that the equivalence classes of *pure* [15] 1-differential infinitesimal deformations are classified by the set $\omega_1 + \hbar \cdot \mathcal{H}^2(\boldsymbol{M})$.

2. Some definitions and results

2.1

Definition 1. An ISP on $(\boldsymbol{G}; \beta_1)$ is a formal deformation in Gerstenhaber's sense [16,17] of the algebra $C^\infty(\boldsymbol{G})[[\hbar]]$, i.e.

$$\varphi * \psi = \varphi \cdot \psi + \sum_{i \geq 1} F_i(\varphi; \psi)\, \hbar^i, \quad \varphi, \psi \in C^\infty(\boldsymbol{G}),$$

where [1,23]:
(1) $(\varphi * \psi) * \xi = \varphi * (\psi * \xi)$;
(2) $F_i, i \geq 1$, are left-invariant bidifferential operators on $\boldsymbol{G}$ such that $F_i(\varphi; 1) = F_i(1; \varphi) = 0$;
(3) $F_1(\varphi; \psi) - F_1(\psi; \varphi) = P(\varphi; \psi)$ where P is the Poisson bracket defined by β_1.

Operator F_i is therefore defined as a left-translation of a unique element in $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})$, also designated by $F_i(x; y)$, in the usual non-commutative polynomial notation [4], where x, y represent the first and second components, respectively, of F_i in $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})$.

2.2

If we consider the element $F(x; y) = 1 + \sum_{i \geq 1} F_i(x; y)\hbar^i \in \mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})[[\hbar]]$ conditions (1)–(3) can be written as follows:
(1′) $F(x + y; z) \cdot F(x; y) = F(x; y + z) \cdot F(y; z)$,
(2′) $F_i(x; 1) = F_i(1; y) = 0$,
(3′) $F_1(x; y) - F_1(y; x) = \Lambda_1(x; y)$,
where the product in (1′) is that of $\mathfrak{A}(\mathfrak{g})^{\otimes 3}[[\hbar]]$, and $F(x + y; z) = (\Delta \otimes I)(F(x; y))$, etc., ($\Delta$ being the usual coproduct in $\mathfrak{A}(\mathfrak{g})$); and $\Lambda_1 \in \mathfrak{g} \wedge \mathfrak{g}$ defines the invariant Poisson structure of $(\boldsymbol{G}; \beta_1)$, i.e. in a given basis of $\mathfrak{g}$, Λ_1 is defined by $(\Lambda_1)^{ab}(\beta_1)_{ac} = \delta^b_c$.

2.3

The associativity condition (1′) is equivalent to the infinite set of conditions:

$$\delta F_m(x; y; z) = \alpha_m(x; y; z), \quad m = 1, 2, 3, \ldots,$$

where $\alpha_1(x; y; z) = 0$,

$$\alpha_m(x; y; z) = \sum_{i+j=m\,;\, i,j \geq 1} [F_i(x + y; z) \cdot F_j(x; y) - F_i(x; y + z) \cdot F_j(y; z)]$$

if $m > 1$, and $\delta : \mathfrak{A}(\mathfrak{g})^{\otimes r} \to \mathfrak{A}(\mathfrak{g})^{\otimes (r+1)}$ is the coboundary operator [5], of the complex $(\mathfrak{A}(\mathfrak{g})^{\otimes}; \delta)$, canonically isomorphic to the subcomplex of the usual Hochschild complex $(\mathcal{C}^\infty(\boldsymbol{G}); \delta)$, whose cochains are invariant bidifferential operators on $\boldsymbol{G}$.

Theorem 2 (Cartier [5]). *Let $C \in \mathcal{Z}^r(\mathfrak{g})$ be an r-cocycle in the complex $(\mathfrak{A}(\mathfrak{g})^{\otimes}; \delta)$. Let AC be the skewsymmetric projection of C. Then:*
(1) $AC \in \mathfrak{g} \wedge \overset{r}{\cdots} \wedge \mathfrak{g}$,
(2) $C = AC + \delta B$ *where* $B \in \mathfrak{A}(\mathfrak{g})^{\otimes (r-1)}$,
(3) $\mathcal{H}^r(\mathfrak{A}(\mathfrak{g})^{\otimes}; \delta) \cong \mathfrak{g} \wedge \overset{r}{\cdots} \wedge \mathfrak{g}$, *and the isomorphism being defined by* $[C] \to AC$.

2.4

Definition 2. Let F', F be two ISPS on $(G; \beta_1)$. We say they are equivalent if there exits $E(x) = 1 + \sum_{i\geq 1} E_i(x)\hbar^i \in \mathfrak{A}(\mathfrak{g})\hbar$ such that

$$F'(x; y) = E^{-1}(x+y) \cdot F(x; y) \cdot E(x) \cdot E(y).$$

The latter equality is equivalent to the infinite set of equalities [16,19,23]

$$F'_k(x; y) - F_k(x; y) + G_k(x; y) = \delta E_k(x; y), \quad k = 1, 2, \ldots,$$

where $G_1(x; y) = 0$; and for $k \geq 2$

$$\begin{aligned} G_k(x; y) &\equiv G(E_1, \ldots, E_{k-1}; F'_1, \ldots, F'_{k-1}; F_1, \ldots, F_{k-1})(x; y) \\ &\equiv \sum_{i+j=k} [E_i(x+y) \cdot F'_j(x; y) - F_i(x; y) \cdot E_j(y) - F_i(x; y) \cdot E_j(x)] \\ &\quad - \sum_{i+j=k} E_i(x) \cdot E_j(y) - \sum_{i+j+l=k} F_i(x; y) \cdot E_j(x) \cdot E_l(y), \end{aligned}$$

where $i, j, l \geq 1$.

2.5

Let $\Lambda_1 \in \mathfrak{g} \wedge \mathfrak{g}$ be the invariant 2-tensor which defines the invariant Poisson structure of the symplectic manifold $(G; \beta_1)$ (in components $(\Lambda_1)^{ab}(\beta_1)_{ac} = \delta^b_c$). Let $\mu : \wedge^r \mathfrak{g} \to \wedge_r \mathfrak{g}$ be the corresponding isomorphism. In the skewsymmetric components

$$\begin{aligned} (\mu(t))_{j_1 \cdots j_r} &= (\beta_1)_{j_1 i_1} \cdots (\beta_r)_{j_r i_r} t^{i_1 \cdots i_r}; \\ (\mu^{-1}(\alpha))^{i_1 \cdots i_r} &= (\Lambda_1)^{j_1 i_1} \cdots (\Lambda_1)^{j_r i_r} \alpha_{j_1 \cdots j_r}. \end{aligned}$$

The Poisson cohomology complex $(\wedge^* \mathfrak{g}; \partial)$, $\partial : \wedge^r \mathfrak{g} \to \wedge^{r+1}\mathfrak{g}$, defined by Λ_1 is: $\partial t = -[t\,; \Lambda_1]_{\mathrm{Sch}}$ where $[\,;\,]_{\mathrm{Sch}}$ is the Schouten bracket [18,20,23]. The relation $\mu \circ (-\partial) = \tilde{\delta} \circ \mu$, where $\tilde{\delta}$ is the coboundary in the complex of the Lie algebra $\mathfrak{g}$, is thereby satisfied. Consequently μ induces an isomorphism [20]

$$\bar{\mu} : \mathcal{H}^r_{\Lambda_1}(\mathfrak{g}) \to \mathcal{H}^r(\mathfrak{g}), \qquad \bar{\mu}([t]) = [\mu(t)].$$

We can prove:

Proposition 1.

(1) *Let $h \in \wedge^2(\mathfrak{g})$. Then*

$$\begin{aligned} \partial h = -[h; \Lambda_1]_{\mathrm{Sch}} = -([h^{12}; \Lambda_1^{13}] + [h^{12}; \Lambda_1^{23}] + [h^{13}; \Lambda_1^{23}] \\ + [\Lambda_1^{12}; h^{13}] + [\Lambda_1^{12}; h^{23}] + [\Lambda_1^{13}; h^{23}]) \end{aligned} \tag{1}$$

where $\Lambda_1^{12} = \Lambda \otimes 1$, $\Lambda_1^{13} = P^{23} \cdot \Lambda_1^{12} \cdot P^{23}$, etc., as in the classical Yang–Baxter notation.

(2) *Let $E \in \mathfrak{g}$, then*

$$\partial E = -[E\,;\,\Lambda_1]_{\mathrm{Sch}} = -([E^1\,;\,\Lambda_1^{12}] + [E^2;\,\Lambda_1^{12}]), \tag{2}$$

where [;] represents a bracket in the algebra $\mathfrak{A}(\mathfrak{g})^{\otimes 3}$ or $\mathfrak{A}(\mathfrak{g})^{\otimes 2}$.

3. An invariant star product on $(G;\,\beta_1)$ determined by the formal cocycle $\beta_\hbar$

3.1

Let $\beta_t = \beta_1 + \beta_2 t + \cdots + \beta_R t^{R-1}$, $t \in \mathbb{R}$ where R is finite but arbitrary and $\beta_i \in \mathcal{Z}^2(\mathfrak{g})$. When t is small, $\tilde{\beta}_t$ is an isomorphism. Let $(G;\,\beta_t)$ be the corresponding symplectic manifold. Let $\bar{\mathfrak{g}}_{\beta_t}$ be the central extension of $\mathfrak{g}$ by the cocycle β_t,

$$\bar{\mathfrak{g}}_{\beta_t} = \mathfrak{g} \oplus \mathbb{R}\mathbf{E}, \qquad [\bar{x};\,\bar{y}] = [x;\,y] + \beta_t(x;\,y)\mathbf{E},$$

where $\bar{x} = x + a\,\mathbf{E}$, $\bar{y} = y + b\,\mathbf{E}$ $(x, y \in \mathfrak{g})$; and $\mathbf{E}$ is a generator. Let $\overline{G}_{\beta_t}$ be the simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}_{\beta_t}$. Let $\bar{\gamma}_{\beta_t}(\bar{x};\,\bar{y})$ be the Campbell–Hausdorff formula for the Lie algebra $\bar{\mathfrak{g}}_{\beta_t}$. The coadjoint orbit of point $\xi + u\,\mathbf{E}^* \in \bar{\mathfrak{g}}^*_{\beta_t}$ is generated by the actions [24]

$$\overline{\mathrm{Ad}}^*(\overline{\exp}\cdot\bar{x})\cdot(\xi + u\,\mathbf{E}^*) = \mathrm{Ad}^*(\exp x)\cdot\xi + u\cdot f_{\beta_t}(-x) + u\,\mathbf{E}^*,$$

where $f_{\beta_t}(x) = \tilde{\beta}_t(x)\cdot A(x) \in \mathfrak{g}^*$ and

$$A(x) = \frac{\exp(\mathrm{ad}\,x) - 1}{\mathrm{ad}\,x}.$$

We are only interested in the orbit of point $\mathbf{E}^* \in \mathfrak{g}^*_{\beta_t}$. Obviously

$$\overline{\mathrm{Ad}}^*(\overline{\exp}\,\xi\cdot\bar{x})\cdot\mathbf{E}^* = f_{\beta_t}(-x) + \mathbf{E}^*.$$

3.2

Let us consider the *formal* expression

$$(\varphi_1 \diamond \varphi_2)(\xi) = \int_{\mathfrak{g}\times\mathfrak{g}} \mathrm{e}^{-2\pi\mathrm{i}\langle(\xi+\mathbf{E}^*);\,\hbar^{-1}\bar{\gamma}(\hbar x;\hbar y)\rangle}(\overline{\mathcal{F}}\varphi_1)(x)\cdot(\overline{\mathcal{F}}\varphi_2)(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{3}$$

where $\mathcal{F}$ is the Fourier transform [30] and φ_1, φ_2 are functions on $\mathfrak{g}^* + \mathbf{E}^*$. In the case of the canonical symplectic manifold $(\mathbb{R}^{2n};\,\beta_1) \equiv (G;\,\beta_1)$ expression (3) is the integral form of Moyal star product obtained from the Weyl quantization on this phase space. In particular, it can be obtained from the integral form of the star product on the non-compact symmetric Khäler orbit $((\mathbb{R}^{2n})^*;\,\Lambda_1)$, of the Heisenberg group H_n obtained from the Berenzin quantization and from operators in the irreducible representation of H_n, determined by the orbit, related to the geodesic symmetry at every point in the above symmetric space [21].

We refer to [27,28] for the construction of deformations and quantum groups in the setting of multiplier algebras and in the more general case of a Lie group and an *abelian subgroup*.

In the general case $(\boldsymbol{G};\beta_1)$ [11] the object is to use the expression (3) to get a star product on the orbit $\mathcal{O}_{\mathbf{E}^*}\subseteq\mathfrak{g}^*+\mathbf{E}^*$ of the Lie group $\bar{\boldsymbol{G}}_{\beta_t}$, which is invariant by the coadjoint action of this group on the orbit.

In [24] we obtained from (3) a formal series

$$\varphi_1\diamond\varphi_2=\varphi_1\cdot\varphi_2+\sum_{R\geq 1}Q_R(\varphi_1;\varphi_2)\hbar^R, \tag{4}$$

which is a formal deformation of $C^\infty(\mathfrak{g}^*+\mathbf{E}^*;\mathbb{R})$, i.e., which satisfies $(\varphi_1\diamond\varphi_2)\diamond\varphi_3=\varphi_1\diamond(\varphi_2\diamond\varphi_3)$, where $\varphi_i\in C^\infty(\mathfrak{g}^*+\mathbf{E}^*;\mathbb{R})$ and Q_R, $R\geq 1$, are bidifferential operators on $\mathfrak{g}^*+\mathbf{E}^*$, invariant by $\overline{\mathrm{Ad}}^*\overline{\boldsymbol{G}}_{\beta_t}$, and such that $Q_R(\varphi_1;1)=Q_R(1;\varphi_1)=0$. Moreover, on the orbit $\mathcal{O}_{\mathbf{E}^*}$ this formal deformation is an invariant star product.

3.3

To obtain [11] an ISP on $\boldsymbol{G}$ from the one on $\mathcal{O}_{\mathbf{E}^*}$ above, we note that $\mathrm{d}f_{\beta_t}(0)=\tilde{\beta}_t$ and so, the mapping

$$\begin{aligned}\exp U_0\equiv U_e\subseteq\boldsymbol{G}&\xrightarrow{\mathcal{K}_t}\mathcal{O}_{\mathbf{E}^*}\subseteq\mathfrak{g}^*+\mathbf{E}^*,\\ \exp x&\longrightarrow\overline{\mathrm{Ad}}^*(\overline{\exp}\,x)\cdot\mathbf{E}^*=f_{\beta_t}(-x)+\mathbf{E}^*\end{aligned} \tag{5}$$

is a local diffeomorphism at $e\in\boldsymbol{G}$. If $W_e\subset U_e$ is a symmetric neighborhood such that $W_e\cdot W_e\subset U_e$, we can prove the commutativity relation

$$(\overline{\mathrm{Ad}}^*\cdot(\overline{\exp}z)\circ\mathcal{K}_t)(y)=(\mathcal{K}_t\circ\lambda_{\exp z})(y),$$

for all $y\in W_e$, for all $z\in W_0=\log W_e$. The mapping $\mathcal{K}_t$ allows us to pull the star product on $\mathcal{O}_{\mathbf{E}^*}$ (4) back to $\boldsymbol{G}$ and by the equivariance shown in the above equality (W_e generates $\boldsymbol{G}$), the star product so obtained is $\boldsymbol{G}$-invariant, and can be written as

$$\psi_1\tilde{*}\psi_2=\psi_1\cdot\psi_2+\sum_{S\geq 1}\tilde{F}_S^t(\psi_1;\psi_2)\hbar^S, \tag{6}$$

where $\psi_1,\psi_2\in C^\infty(\boldsymbol{G};\mathbb{R})$ and

$$\tilde{F}_R^t(\psi_1;\psi_2)(\exp x)=Q_R(\psi_1\circ\mathcal{K}_t^{-1};\psi_2\circ\mathcal{K}_t^{-1})(\xi),$$

$\xi=\mathcal{K}_t(\exp x)$, $x\in W_0\subset\mathfrak{g}$. $\tilde{F}_R^t$ is therefore an invariant bidifferential operator on $\boldsymbol{G}$ and is defined then as a left-translation of an element $F_R^t(x;y)\in\mathfrak{A}(\mathfrak{g})\otimes\mathfrak{A}(\mathfrak{g})$.

3.4

Operator $F_R^t(x;y)$ depends on t through a polynomial in the components of β_t and Λ_t. By expanding this product of a *finite* number of analytic functions of t (with t small), we get

$$F_S^t(x;y)=\sum_{L\geq 0}F_{S\,L}(x;y)\,t^L,\quad S\geq 1$$

and also $F_{SL}(x;0) = F_{SL}(0;y) = 0$. If we now set $t = \hbar$ in (6), we prove [24]:

Theorem 3. *Let*

$$F(x;y) = 1 + \sum_{R\geq 1} F_R(x;y)\,\hbar^R, \quad \textit{where } F_R(x;y) = \sum_{S+T=R} F_{ST}(x;y).$$

$F(x;y)$ is then an invariant star product on the symplectic manifold $(\mathbf{G};\beta_1)$. We will say it is determined by the cocycle

$$\beta_\hbar = \beta_1 + \beta_2\,\hbar + \beta_3\,\hbar^2 + \cdots + \beta_R\,\hbar^{R-1} + \cdots$$

3.5

Let $\{e_a;\ a = 1, 2, \ldots, 2n\}$ be a basis of $\mathfrak{g}$ and $\{x_a\}$ the corresponding canonical coordinates. Then

$$\left(\frac{\partial}{\partial x^a}\right)(x) = A(-x)_a^j \bar{e}_j(x), \quad \text{where } \bar{e}_j(x) = T_e\lambda_{\exp x}\cdot e_j, \tag{7}$$

whereby we obtain the following Poisson brackets:

$$\begin{aligned} Q_1^t(\varphi_1;\varphi_2)(\xi+\mathbf{E}^*) &= \frac{1}{2}\left(\xi_k c_{i\,j}^k + (\beta_t)_{i\,j}\right)\cdot\frac{\partial\varphi_1}{\partial\xi_i}(\xi+\mathbf{E}^*)\cdot\frac{\partial\varphi_2}{\partial\xi_j}(\xi+\mathbf{E}^*),\\ F_1^t(\psi_1;\psi_2) &= \tfrac{1}{2}\Lambda_t^{a\,b}(\bar{e}_a\psi_1)(0)\cdot(\bar{e}_b\psi_2)(0)\\ F_1(\psi_1;\psi_2) &= \tfrac{1}{2}\Lambda_1^{a\,b}(\bar{e}_a\psi_1)(0)\cdot(\bar{e}_b\psi_2)(0), \end{aligned}$$

where $c_{i\,j}^k$ are structure constants of $\mathfrak{g}$.

Thus,

$$F_1^t(x;y) = \tfrac{1}{2}\Lambda_t^{a\,b} e_a\otimes e_b, \qquad F_1(x;y) = \tfrac{1}{2}\Lambda_1^{a\,b} e_a\otimes e_b.$$

3.6

The next two propositions help to prove the last theorem in this section, which is the first step in proving Drinfeld's classification theorem (Theorem 1). The proofs are in [24].

Proposition 2. *Let F, $\bar{F}$ be the star products on $(\mathbf{G};\beta_1)$ determined, respectively, by the formal cocycles*

$$\begin{aligned} \beta_\hbar &= \beta_1 + \beta_2\,\hbar + \cdots + \beta_{R-1}\,\hbar^{R-2} + 0\,\hbar^{R-1} + \beta_{R+1}\,\hbar^R + \cdots\\ \bar{\beta}_\hbar &= \beta_1 + \beta_2\,\hbar + \cdots + \beta_{R-1}\,\hbar^{R-2} + \bar{\beta}_R\,\hbar^{R-1} + \bar{\beta}_{R+1}\,\hbar^R + \cdots \end{aligned}$$

Then

$$F_1 = \bar{F}_1;\ldots;F_{R-1} = \bar{F}_{R-1} \quad \textit{and} \quad \bar{F}_R - F_R = -\tfrac{1}{2}\mu^{-1}(\bar{\beta}_R).$$

Proposition 3. *Let F, $\bar{F}$ be star products on $(\boldsymbol{G}; \beta_1)$ which coincide to the order $(R-1)$, i.e.,*

$$F_1 = \bar{F}_1; \ldots ; F_{R-1} = \bar{F}_{R-1}.$$

Then, there exist $h_R \in \mathfrak{g} \wedge \mathfrak{g}$ and $E_R \in \mathfrak{A}(\mathfrak{g})$ such that

$$\bar{F}_R - F_R = h_R + \tilde{\delta} E_R.$$

Moreover, h_R is not only a Hochschild 2-cocycle but also a Poisson 2-cocycle, i.e. $\partial h_R = 0$, where ∂h_R is given by expression (1). $\mu(h_R)$ is then a 2-cocycle on $\mathfrak{g}$.

From these propositions, we can prove by induction:

Theorem 4. *Any star product F on $(\boldsymbol{G}; \beta_1)$ determines a formal 2-cocycle on the Lie algebra $\mathfrak{g}$,*

$$\beta_\hbar = \beta_1 + \beta_2 \hbar + \cdots + \beta_R \hbar^{R-1} + \cdots$$

such that F is equivalent to the star product determined by $\beta_\hbar$ as in Theorem 3.

4. Star products on $\mathfrak{g}$, determined respectively by formal cocycles $\beta_\hbar$ and $\omega_\hbar = \beta_\hbar + \tilde{\delta}\alpha_\hbar$, are equivalent

4.1

Let

$$\beta_t = \beta_1 + \beta_2 t + \cdots + \beta_R t^{R-1}$$

and

$$\omega_t = \beta_1 + (\beta_2 + \tilde{\delta}\alpha_2) t + \cdots + (\beta_R + \tilde{\delta}\alpha_R) t^{R-1} \equiv \beta_t + \tilde{\delta}\alpha_t$$

be cocycles where $\alpha_i \in \mathfrak{g}^*$ and $\tilde{\delta}$ is the de Rham coboundary operator on the invariant complex on $\boldsymbol{G}$. Let $\overline{\mathfrak{g}}_{\beta_t}$, $\overline{\mathfrak{g}}_{\omega_t}$ be the central extensions of $\mathfrak{g}$ corresponding respectively to β_t and ω_t. Let $\overline{\boldsymbol{G}}_{\beta_t}$ and $\overline{\boldsymbol{G}}_{\omega_t}$ be corresponding, connected, and simply connected Lie-groups. $\overline{\mathfrak{g}}_{\beta_t}$ and $\overline{\mathfrak{g}}_{\omega_t}$ are isomorphic. Consider the mapping

$$\begin{aligned} \lambda_{\alpha_t} : \mathfrak{g}^* + \mathbf{E}^* &\longrightarrow \mathfrak{g}^* + \mathbf{E}^* \\ \xi + \mathbf{E}^* &\longrightarrow \xi + \alpha_t + \mathbf{E}^*. \end{aligned}$$

We can prove:

Proposition 4.
(1) $\overline{\mathrm{Ad}}^* \cdot (\overline{\exp}_{\omega_t} x) \circ \lambda_{\alpha_t} = \lambda_{\alpha_t} \circ (\overline{\mathrm{Ad}}^* \cdot (\overline{\exp}_{\beta_t} x))$ *for all $x \in \mathfrak{g}$.*

(2) *The operators* $Q_R^{\omega_t}$, $Q_R^{\beta_t}$, $R \geq 1$, *defining the formal deformations on* $\mathfrak{g}^* + \mathbf{E}^*$ *determined, respectively, by* ω_t *and* β_t *are related as follows:*

$$Q_R^{\omega_t}\left(\frac{\partial}{\partial \xi}; \frac{\partial}{\partial \xi}; \xi\right) = Q_R^{\beta_t}\left(\frac{\partial}{\partial \xi}; \frac{\partial}{\partial \xi}; \xi - \alpha_t\right) \quad \text{for all } \xi \in \mathfrak{g}^*.$$

(3) $\varphi_1 \diamond_{\omega_t} \varphi_2 = ((\varphi_1 \circ \lambda_{\alpha_t}) \diamond_{\beta_t} (\varphi_2 \circ \lambda_{\alpha_t})) \circ \lambda_{-\alpha_t}, \quad \varphi_1, \varphi_2 \in C^\infty(\mathfrak{g}^*; \mathbb{R})$.

4.2

Let $\mathcal{K}_{\beta_t}$, $\mathcal{K}_{\omega_t}$ be local diffeomorphisms (5) corresponding, respectively, to cocycles β_t and ω_t. Let $\psi_1, \psi_2 \in C^\infty(\boldsymbol{G}; \mathbb{R})$. From (3) in Proposition 4 and definitions (4) and (6) of star product on $(\boldsymbol{G}; \omega_t)$ and $(\boldsymbol{G}; \beta_t)$ determined, respectively, by cocycles ω_t and β_t we get the relation

$$(\psi_1 \tilde{*}_{\omega_t} \psi_2) \circ M_t = ((\psi_1 \circ M_t) \tilde{*}_{\beta_t} (\psi_2 \circ M_t)), \tag{8}$$

where $M_t = \mathcal{K}_{\omega_t}^{-1} \circ \lambda_{\alpha_t} \circ \mathcal{K}_{\beta_t}$ with M_t defined in a neighborhood of $e \in \boldsymbol{G}$ and t small.

Let $\psi_1 *_{\omega_\hbar} \psi_2$, $\psi_1 *_{\beta_\hbar} \psi_2$ be the star products on $(\boldsymbol{G}; \beta_1)$ determined as in Theorem 3, respectively, from cocycles $\omega_\hbar$ and $\beta_\hbar$. By expansion of both sides of expression (8) with $t = \hbar$ the equality of both series in powers of $\hbar$ allows us to prove [24]:

Theorem 5. *Let* $F^{\omega_\hbar} = 1 + \sum_{R \geq 1} F_R^{\omega_\hbar} \hbar^R$, $F^{\beta_\hbar} = 1 + \sum_{R \geq 1} F_R^{\beta_\hbar} \hbar^R$ *be star products on* $(\boldsymbol{G}; \beta_1)$ *determined, respectively, by cocycles* $\omega_\hbar = \beta_\hbar + \tilde{\delta}\alpha_\hbar$ *and* $\beta_\hbar$. *There must then be an element*

$$L(x) = 1 + \sum_{R \geq 1} L_R \hbar^R \in \mathfrak{A}(\mathfrak{g})[[\hbar]]$$

defined from (8) *such that*

$$F^{\omega_\hbar}(x; y) = (L^{-1}(x + y)) \cdot F^{\beta_\hbar}(x; y) \cdot L(x) \cdot L(y). \tag{9}$$

That is, $F^{\omega_\hbar}$ *and* $F^{\beta_\hbar}$ *are equivalent by* $L(x)$.

5. Proving the converse of Theorem 5

5.1

Next two propositions will be useful in Section 5.2.

Proposition 5. *Let* F', F'' *be the invariant star products on* $(\boldsymbol{G}; \beta_1)$ *determined, respectively, by cocycles*

$$\beta'_\hbar = \beta_1 + \beta_2 \hbar + \cdots + \beta_{R-1} \hbar^{R-2} + \beta'_R \hbar^{R-1} + \cdots$$
$$\beta''_\hbar = \beta_1 + \beta_2 \hbar + \cdots + \beta_{R-1} \hbar^{R-2} + \beta''_R \hbar^{R-1} + \cdots$$

The following equalities are then verified ($R \geq 2$)

$$F'_1 = F''_1;\; F'_2 = F''_2;\ldots;F'_{R-1} = F''_{R-1} \quad \textit{and} \quad F''_R - F'_R = -\tfrac{1}{2}\mu^{-1}(\beta''_R - \beta'_R).$$

Proof. Let F be the star product determined by cocycle

$$\beta_\hbar = \beta_1 + \beta_2\hbar + \cdots + \beta_{R-1}\hbar^{R-2} + 0\hbar^{R-1} + \beta'_{R+1}\hbar^R + \cdots$$

The result follows from Proposition 2 in cases F', F and F'', F. □

Proposition 6. *Let F, F' be star products on $(\boldsymbol{G};\beta_1)$ determined, respectively, by cocycles*

$$\beta_\hbar = \beta_1 + \beta_2\hbar + \cdots + \beta_R\hbar^{R-1} + \beta_{R+1}\hbar^R + \cdots$$
$$\beta'_\hbar = \beta_1 + \beta_2\hbar + \cdots + (\beta_R + \tilde{\delta}\alpha_R)\hbar^{R-1} + \beta_{R+1}\hbar^R + \cdots$$

That is, $\beta'_\hbar$ and $\beta_\hbar$ differ only by an exact 2-cocycle $\tilde{\delta}\alpha_R$ at the coefficient of $\hbar^{R-1}$. In this case, element $L(x)$ in the equivalence between F' and F of Theorem 5,

$$F'(x;y) = (L^{-1}(x+y))\cdot F(x;y)\cdot L(x)\cdot L(y),$$

verifies:

$$L_1(x) = \cdots = L_{R-2}(x) = 0 \quad (R \geq 3) \quad \text{and} \quad L_{R-1} = \mu^{-1}(\alpha_R) \quad (R \geq 2).$$

Proof.

(a) Let

$$\Lambda'_\hbar = \sum_{i\geq 1}\Lambda'_i\hbar^{i-1}, \qquad \Lambda_\hbar = \sum_{i\geq 1}\Lambda_i\hbar^{i-1}$$

be formal series defined through the relations $(\Lambda'_\hbar)^{ab}(\beta'_\hbar)_{ac} = \delta^b_c$ and $(\Lambda_\hbar)^{ab}(\beta_\hbar)_{ac} = \delta^b_c$. Then, $\Lambda'_i = \Lambda_i$; $1 \leq i \leq R-1$ and $(\Lambda_1)^{ab}(\beta_1)_{ac} = \delta^b_c$. If $\beta'_\hbar$ and $\beta_\hbar$ are polynomials of arbitrary degree, these series are convergent when $\hbar$ is small.

(b) The invariant operators L_S are defined, (8) and (9), in the following Taylor expansion at point $(\hbar = 0; x)$ as

$$\psi_1(y) = (\psi_1 \circ M_\hbar)(x) = \psi_1(x) + \sum_{A\geq 1}(L_S\psi_1)(x)\hbar^S,$$

where

$$y = M_\hbar(x) = ((\mathcal{K}_{\omega_\hbar})^{-1} \circ \lambda_{\alpha_\hbar} \circ \mathcal{K}_{\beta_\hbar})(x),$$

and $\{x\}$ are cannonical coordinates at $e \in \boldsymbol{G}$.

This function is analytical (Sections 3.1 and 3.3) in a neighborhood of the point $(\hbar = 0; x = 0)$. Hence the series expansion

$$y^a = \sum_{k\geq 0} A^a_k(x)\hbar^k, \quad A^k_0(x) = x^a, \qquad a = 1,\ldots,2n,$$

is convergent and $A_k^a(x)$ is analytical in that neighborhood. We can also write the convergent series

$$(y^a - x^a)^l = \sum_{S\geq 1} \Omega_S^{a\,l}\,\hbar^S,$$

where

$$\Omega_S^{a\,l}(x) = \sum_{k_1+\cdots+k_l=S,\,k_j\geq 1} A_{k_1}^a(x)\cdots A_{k_l}^a(x), \quad 1\leq l\leq S. \tag{10}$$

Finally we obtain ($S\geq 1$)

$$(L_S\psi_1)(x) = \sum_{S_1+\cdots+S_{2n}=S} \frac{1}{L!}\Omega_{S_1}^{1\,l_1}(x)\cdots\Omega_{S_{2n}}^{2n\,l_{2n}}(x)\cdot\frac{\partial^{l_1+\cdots+l_{2n}}\psi_1}{(\partial x^1)^{l_1}\cdots(\partial x^{2n})^{l_{2n}}}(x), \tag{11}$$

for $S_i\geq l_i\geq 0$, $l_1+\cdots+l_{2n}\geq 1$. Also if $l_a=0$ *then* S_a *must be zero*. Let us define $\Omega_0^{j\,0}(x)=1$.

(c) To compute elements $L_S(x)\in\mathfrak{A}(\mathfrak{g})$, $1\leq S\leq R-1$, we need to compute $A_k^a(0)$ for $1\leq k\leq R-1$, $a=1,\ldots,2n$.

– The equations of $\eta=\mathcal{K}_{\beta_t}(x)$, (see Sections 3.1 and 3.3) are

$$\eta_a = -x^i(\beta_\hbar)_{i\,a} - x^i(\beta_\hbar)_{i\,k}\sum_{r\geq 1} M_{i_1\cdots i_r;a}^k x^{i_1}\cdots x^{i_r},$$

where constants $M_{i_1\cdots i_r;a}^k$ are homogeneous polynomials in structure constants of $\mathfrak{g}$ with respect to the given basis in $\mathfrak{g}$.

– From the definition of λ_{α_t}, we have

$$\xi = \lambda_{\alpha_R\hbar^{R-1}}(\eta) = \eta + \alpha_R\,\hbar^{R-1}.$$

– The equations of $y=(\mathcal{K}_{\beta'_\hbar})^{-1}(x)$ can be written as

$$y^b = (\Lambda'_\hbar)^{a\,b}\xi_a + y^i\cdot(\Lambda'_\hbar)^{a\,b}(\beta'_\hbar)_{i\,k}\sum_{r\geq 1} M_{i_1\cdots i_r;a}^k y^{i_1}\cdots y^{i_r}.$$

The equations of $y=M_\hbar(x)$ are then obtained by composing the above mappings. In particular, at point $x=0$, we get

$$y^b(0) = (\Lambda'_\hbar)^{a\,b}(\alpha_R)_a\hbar^{R-1} + y^j(0)\cdot(\Lambda'_\hbar)^{a\,b}\cdot(\beta'_\hbar)_{j\,k}\cdot\sum_{r\geq 1} M_{i_1\cdots i_r;a}^k y^{i_1}(0)\cdots y^{i_r}(0).$$

From the expression $y=M_\hbar(x)$,

$$y^i(0) = \sum_{k\geq 1} A_k^i(0)\,\hbar^k \quad (i=1,\ldots,2n).$$

From the equality of the two preceding series in powers of $\hbar$, we get

$$\begin{aligned} &A_1^i(0) = A_2^i(0) = \cdots = A_{R-2}^i(0) = 0, \quad i = 1, 2, \ldots, 2n, \\ &A_{R-1}^i(0) = (\Lambda_1')^{a\,i}(\alpha_R)_a = (\Lambda_1)^{a\,i}(\alpha_R)_a = \mu^{-1}(\alpha_R)^i. \end{aligned} \tag{12}$$

(d) Assuming $1 \le S \le R-2$ in (10), then $1 \le k_j \le R-2$ and $1 \le l \le S$. Thus, from (12) we get $\Omega_S^{a\,l}(0) = 0$ for $1 \le S \le R-2$ and $1 \le l \le S$. Hence

$$(L_1)_{x=0} = (L_2)_{x=0} = \cdots = (L_{R-2})_{x=0} = 0.$$

Obviously, in the polynomial notation in $\mathfrak{A}(\mathfrak{g})$, this is equivalent to

$$0 = (L_i)_{x=0} \equiv L_i(x) \in \mathfrak{A}(\mathfrak{g}), \quad i = 1, 2, \ldots, R-2.$$

(e) We now compute the operator $(L_{R-1})_{x=0}$. When $S = R-1 \ge 1$, expression (11) contains all the terms $\Omega_{S_j}^{j\,l_j}(0)$ where $1 \le l_j \le S_j \le R-1$. From (12) all these terms are zero except $\Omega_{R-1}^{j\,l_j}(0)$, $1 \le l_j \le S_j = R-1$. Hence, necessarily

$$(L_{R-1}\psi_1)(0) = \sum_{l_j \ge 1} \Omega_{R-1}^{j\,l_j}(0)\left(\frac{\partial^{l_j}}{(\partial x^j)^{l_j}}\psi_1\right)(0).$$

Also from (10) and (12) it must be $l_j = 1$ for $j = 1, \ldots, 2n$. So

$$\Omega_{R-1}^{j\,1}(0) = A_{R-1}^j(0) = \mu^{-1}(\alpha_R)^j.$$

Hence

$$(L_{R-1})_{x=0} = \mu^{-1}(\alpha_R)^j \cdot \left(\frac{\partial}{\partial x^j}\right)_{x=0} = \mu^{-1}(\alpha_R)^j \cdot \bar{e}_j(0)$$

and the corresponding element in $\mathfrak{A}(\mathfrak{g})$ is therefore

$$L_{R-1}(x) = \mu^{-1}(\alpha_R)^j \cdot e_j = \mu^{-1}(\alpha_R).$$

The proof is now complete. □

5.2

We now can prove the converse of Theorem 5.

Theorem 6. *Let F, F' be the ISPS on $(\mathbf{G}; \beta_1)$ determined, respectively, by the cocycles*

$$\begin{aligned} \beta_\hbar &= \beta_1 + \beta_2\hbar + \cdots + \beta_R\hbar^{R-1} + \cdots \\ \beta_\hbar' &= \beta_1 + \beta_2'\hbar + \cdots + \beta_R'\hbar^{R-1} + \cdots \end{aligned}$$

Assuming these products to be equivalent, there must be $\alpha_2, \ldots, \alpha_R \ldots \in \mathfrak{g}^*$ such that

$$\beta_i' - \beta_i = \tilde{\delta}\alpha_i, \quad i = 2, 3, \ldots, R, \ldots$$

Proof. Let $E(x) = 1 + \sum_{i\geq 1} E_i(x)\hbar^i \in \mathfrak{A}(\mathfrak{g})[[\hbar]]$ be the element defining the equivalence

$$F'(x;y) = (E(x+y))^{-1} \cdot F(x;y) \cdot E(x) \cdot E(y). \tag{a}$$

(1) $\Lambda'_1 = \Lambda_1$ and from Section 3.5, we have

$$F'_1(x;y) = F_1(x;y) = \tfrac{1}{2}\Lambda_1^{ab} e_a \otimes e_b \equiv \tfrac{1}{2}\Lambda_1(x;y). \tag{b}$$

Thus $F_1(x;y) - F_1(y;x) = \Lambda_1(x;y)$.

The term $\hbar$ in equivalence (a) is

$$F'_1(x;y) - F_1(x;y) = \delta E_1(x;y). \tag{c}$$

From (b) and (c) we get $\delta E_1(x;y) = 0$. Thus, $E_1(x) \in \mathfrak{g}$ from Theorem 2 ($\partial\mathbb{R} \equiv \partial\mathfrak{A}(\mathfrak{g})^{\otimes 0} = 0$), and therefore

$$E_1(x+y) = E_1(x) + E_1(y). \tag{d}$$

By Proposition 5, for $R = 2$ and star products F and F', we can write

$$F'_1 = F_1 \quad \text{and} \quad F'_2 - F_2 = -\tfrac{1}{2}\mu^{-1}(\beta'_2 - \beta_2). \tag{e}$$

Also, the term $\hbar^2$ in equivalence (a) is (see Definition 2)

$$F'_2 - F_2 + G_2(E_1; F_1; F'_1) = \delta E_2. \tag{f}$$

From (e) and (f), we get

$$G_2(E_1; F_1; F'_1) = \tfrac{1}{2}\mu^{-1}(\beta'_2 - \beta_2) + \delta E_2,$$

whose skewsymmetric projection is

$$AG_2(E_1; F_1; F'_1) = \tfrac{1}{2}\mu^{-1}(\beta'_2 - \beta_2). \tag{g}$$

We can also compute the left-hand side of (g) from Definition 2, allowing for (b), (d) and (2). We then obtain

$$\begin{aligned} G_2(E'_1 F_1; F'_1) &= [E_1(x); F_1(x;y)] - [F_1(x;y); E_1(y)] - E_1(x)E_1(y) \\ &= \tfrac{1}{2}[E_1(x); \Lambda_1(x;y)]_{\mathrm{Sch}} - E_1(x)E_1(y) \\ &= -\tfrac{1}{2}\partial E_1(x;y) - E_1(x)E_1(y). \end{aligned}$$

Thus,

$$AG_2(E_1; F_1; F'_1) = -\tfrac{1}{2}\partial E_1(x;y). \tag{h}$$

From (g) and (h)

$$\mu^{-1}(\beta'_2 - \beta_2) = -\partial E_1$$

and so

$$\beta'_2 - \beta_2 = \mu(-\partial E_1) = \tilde{\delta}(\mu(E_1)) \quad \text{or} \quad \beta'_2 = \beta_2 + \tilde{\delta}\alpha_2,$$

where we set $\alpha_2 = \mu(E_1)$.

We have thereby proved the theorem for $i = 2$.

To proceed by induction, we eliminate $\tilde{\delta}\alpha_2$ as follows. Let $F^{(2)}$ be the star product determined by cocycle

$$\beta_\hbar^{(2)} = \beta_1 + \beta_2 \hbar + \beta_3' \hbar^2 + \cdots + \beta_R' \hbar^{R-1} + \cdots \tag{k}$$

Compare it with F' determined by cocycle $\beta_\hbar'$. Proposition 6 for F', $F^{(2)}$ and $R = 2$ allows us to write

$$F'(x; y) = (L^{(2)}(x + y))^{-1} \cdot F^{(2)}(x; y) \cdot L^{(2)}(x) \cdot L^{(2)}(y), \tag{l}$$

where

$$L_1^{(2)}(x) = \mu^{-1}(\alpha_2). \tag{m}$$

From (a) and (l) we get

$$F^{(2)}(x; y) = (E^{(2)}(x + y))^{-1} \cdot F(x; y) \cdot E^{(2)}(x) \cdot E^{(2)}(y), \tag{m$'$}$$

where we have defined $E^{(2)} = E \cdot (L^{(2)})^{-1}$. So $F^{(2)}$ and F are thus equivalent by $E^{(2)}$. Moreover, from $E^{(2)} \cdot L^{(2)} = E$, at order $\hbar^1$ we get $E_1^{(2)} + L_1^{(2)} = E_1$. From (m) and $\alpha_2 = \mu(E_1)$, we then obtain

$$E_1^{(2)}(x) = 0.$$

We have thus proved that $\beta_2' = \beta_2 + \tilde{\delta}\alpha_2$, and star products $F^{(2)}$, F are equivalent, (m$'$)where $E_1^{(2)} = 0$.

(2) The second step in proof by induction is to treat star produts $F^{(2)}$, F as we did F', F. We thereby prove that $E_2^{(2)} \in \mathfrak{g}$, $\beta_3' = \beta_3 + \tilde{\delta}\alpha_3$ where $\alpha_3 = \mu(E_2^{(2)})$, and that star product $F^{(3)}$ determined by cocycle

$$\beta_\hbar^{(3)} = \beta_1 + \beta_2 \hbar + \beta_3 \hbar^2 + \beta_4' \hbar^3 + \cdots$$

and F are equivalent,

$$F^{(3)}(x; y) = (E^{(3)}(x + y))^{-1} \cdot F(x; y) \cdot E^{(3)}(x) \cdot E^{(3)}(y),$$

where $E_1^{(3)}(x) = E_2^{(3)}(x) = 0$. The theorem is thus proved for $i = 3$ and we proceed to the third step.

$(R-1)$ Suppose we have proved that $\beta_i' - \beta_i = \tilde{\delta}\alpha_i$ for $2 \le i \le R-1$ and that the star product $F^{(R-1)}$ determined by cocycle

$$\beta_\hbar^{(R-1)} = \beta_1 + \beta_2 h + \cdots + \beta_{R-1} \hbar^{R-2} + \beta_R' \hbar^{R-1} + \beta_{R+1}' \hbar^R + \cdots$$

and F are equivalent,

$$F^{(R-1)}(x; y) = (E^{(R-1)}(x; y))^{-1} \cdot F(x; y) \cdot E^{(R-1)}(x) \cdot E^{(R-1)}(y), \tag{n}$$

where $E^{(R-1)} \in \mathfrak{A}(\mathfrak{g})[[\hbar]]$ is such that

$$E_1^{(R-1)} = E_2^{(R-1)} = \cdots = E_{R-2}^{(R-1)} = 0. \tag{p}$$

We need to prove:

(i) $E_{R-1}^{(R-1)} \in \mathfrak{g}$ and so $E_{R-1}^{(R-1)}(x+y) = E_{R-1}^{(R-1)}(x) + E_{R-1}^{(R-1)}(y)$.

(ii) $\beta'_R = \beta_R + \tilde{\delta}\alpha_R$, where $\alpha_R = \mu(E_{R-1}^{(R-1)}) \in \mathfrak{g}^*$.

(iii) $F^{(R-1)}(x;y)$ and $F^{(R)}(x;y)$ determined by cocycle

$$\beta_\hbar^{(R)} = \beta_1 + \beta_2\hbar + \cdots + \beta_R\hbar^{R-1} + \beta'_{R+1}\hbar^R + \cdots$$

are equivalent, i.e.,

$$F^{(R-1)}(x;y) = (L^{(R)}(x+y))^{-1} \cdot F^{(R)}(x;y) \cdot L^{(R)}(x) \cdot L^{(R)}(y),$$

where $L^{(R)}$ satisfies

$$L_1^{(R)} = L_2^{(R)} = \cdots = L_{R-2}^{(R)} = 0 \quad \text{and} \quad L_{R-1}^{(R)} = \mu^{-1}(\alpha_R).$$

(iv) $F^{(R)}$ is equivalent to F

$$F^{(R)}(x;y) = (E^{(R)}(x+y))^{-1} \cdot F(x;y) \cdot E^{(R)}(x) \cdot E^{(R)}(y),$$

where

$$E^{(R)} = E^{(R-1)} \cdot (L^{(R)})^{-1}$$

and this $E^{(R)}$ satisfies

$$E_1^{(R)} = \cdots = E_{R-1}^{(R)} = 0.$$

proof of (i) For star products F and $F^{(R-1)}$ Proposition 5 allows us to write

$$F_1 = F_1^{(R-1)}, \ldots, F_{R-1} = F_{R-1}^{(R-1)} \quad \text{and} \quad F_R^{(R-1)} - F_R = -\tfrac{1}{2}\mu^{-1}(\beta'_R - \beta_R). \tag{q}$$

The term $\hbar^{(R-1)}$ in equivalence (n) is (see Section 2.5)

$$F_{R-1}^{(R-1)} - F_{R-1} + G_{R-1}^{(R-1)}(E_1^{(R-1)}, \ldots, E_{R-2}^{(R-1)}; F_1^{(R-1)}, \ldots, F_{R-2}^{(R-1)}; F_1, \ldots, F_{R-2}) = \delta E_{R-1}^{(R-1)}.$$

From (p) and (q) and Definition 2 we get that the left-hand side of this equation is 0. Then $\delta E_{R-1}^{(R-1)} = 0$ and so $E_{R-1}^{(R-1)} \in \mathfrak{g}$. Hence

$$E_{R-1}^{(R-1)}(x+y) = E_{R-1}^{(R-1)}(x) + E_{R-1}^{(R-1)}(y). \tag{r}$$

proof of (ii) The term $\hbar^R$ in equivalence (n) is

$$F_R^{(R-1)} - F_R + G_R^{(R-1)}(E_1^{(R-1)}, \ldots, E_{R-1}^{(R-1)}; F_1^{(R-1)}, \ldots, F_{R-1}^{(R-1)}; F_1, \ldots, F_{R-1})$$
$$= \delta E_R^{R-1}. \qquad \text{(s)}$$

From (q), (p), (r) and (2),

$$\begin{aligned} G_R^{(R-1)}&(E_1^{(R-1)}, \ldots, E_{R-1}^{(R-1)}; F_1^{(R-1)}, \ldots, F_{R-1}^{(R-1)}; F_1, \ldots, F_{R-1}) \\ &= \tfrac{1}{2}([E_{R-1}^{(R-1)}(x) ; \Lambda_1(x; y)] - [\Lambda_1(x; y) ; E_{R-1}^{(R-1)}(y)]) \\ &= \tfrac{1}{2}[E_{R-1}^{(R-1)}(x) ; \Lambda I_1(x; y)]_{\text{Sch}} = -\tfrac{1}{2}\partial E_{R-1}^{(R-1)}(x; y). \end{aligned}$$

From this equality, the second equality in (q) and (s), we get (recalling that $\delta E_R^{(R-1)}$ is a symmetrical tensor)

$$\beta_R' - \beta_R = \mu(-\partial E_{R-1}^{(R-1)}) = \tilde{\delta}(\mu(E_{R-1}^{(R-1)})) = \tilde{\delta}\alpha_R,$$

where we have defined $\alpha_R = \mu(E_{R-1}^{(R-1)})$. This proves (ii).

proof of (iii) By Proposition 6, for $F^{(R)}$ and $F^{(R-1)}$, there must be $L^{(R)} \in \mathfrak{A}(\mathfrak{g})[[\hbar]]$ such that

$$F^{(R-1)}(x; y) = (L^{(R)}(x + y))^{-1} \cdot F^{(R)}(x; y) \cdot L^{(R)}(x) \cdot L^{(R)}(y), \qquad \text{(t)}$$

where

$$L_1^{(R)} = \cdots = L_{R-2}^{(R)} = 0 \quad \text{and} \quad L_{R-1}^{(R)} = \mu^{-1}(\alpha_R). \qquad \text{(u)}$$

This proves (iii).

proof of (iv) From equivalence (t) and (n),

$$F^{(R)}(x; y) = (E^{(R)}(x + y))^{-1} \cdot F(x; y) \cdot E^{(R)}(x) \cdot E^{(R)}(y),$$

where we have defined $E^{(R)} = E^{(R-1)} \cdot (L^{(R)})^{-1}$. The term $\hbar^i$ of this equality is

$$E_1^{(R)} + L_1^{(R)} = E_1^{(R-1)}, \qquad \text{(v)}$$

$$E_i^{(R)} + L_i^{(R)} + \sum_{j+k=i\,;\ j,k\geq 1} E_j^{(R)} \cdot L_k^{(R)} = E_i^{(R-1)} \quad (i \geq 2). \qquad \text{(w)}$$

From (p), (u) and (v) we get

$$E_1^{(R)} = 0. \qquad \text{(z)}$$

From (w) for $i = 2$

$$E_2^{(R)} + L_2^{(R)} + E_1^{(R)} \cdot L_1^{(R)} = E_1^{(R-1)}.$$

Allowing for (p) and (u) we now get $E_2^{(R)} = 0$. By proceeding in this way, we obtain

$$E_1^{(R)} = \cdots = E_{R-2}^{(R)} = 0$$

and also from (w), for $i = R - 1$,

$$E^{(R)}_{R-1} + L^{(R)}_{R-1} = E^{(R-1)}_{R-1}$$

and from (u) and definition $\alpha_R = \mu(E^{(R-1)}_{R-1})$, we get $E^{(R)}_{R-1} = 0$.

This proves (iv), and the proof of the theorem is now complete. □

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Deformation theory and quantization I and II, Ann. of Phys. 111 (1978) 61–151.

[2] M. Bertelson, M. Cahen and S. Gutt, Equivalence of star products, Université Libre de Bruxelles, Travaux de Mathématiques (1996) 1–15.

[3] P. Bonneau, M. Flato, M. Gerstenhaber and G. Pinczon, The hidden group structure of quantum groups: Strong duality, rigidity and preferred deformations, Comm. Math. Phys. 161 (1994) 125–156.

[4] N. Bourbaki, *Algèbre,* Chapitres 4 à 7 (Masson, Paris, 1981).

[5] P. Cartier, *Hyperalgèbres et groupes de Lie formels,* Séminaire Sophus Lie, 2ème année Faculté des Sciences de Paris (1981).

[6] A. Connes, M. Flato and D. Sternheimer, Closed star products an cyclic cohomology, Lett. Math. Phys. 24 (1992) 1.

[7] P. Deligne, Déformations de l'Algèbre des Fonctions d'une Variété Symplectique: Comparaison entre Fedosov et De Wilde, Lecomte; Selecta Math., New Series 1 (1995) 667–697.

[8] M. De Wilde and P. Lecomte, Existence of star-products and of formal deformations in Poisson Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. 7 (1983) 487–496.

[9] M. De Wilde and P. Lecomte, Existence of star-products revisited; Note di Mathematica X (Suppl. 1) (1990) 205–216.

[10] M. De Wilde and P. Lecomte, Existence et classification des star-produits sur les variétés symplectiques. *Notes du cours dans l'Ecole d'Eté du C.I.M.P.A., Quantification – Quantification par déformation* (Nice, France, 1996).

[11] V.G. Drinfeld, On constant, quasiclassical solutions of the Yang–Baxter quantum equation, Soviet Math. Dokl. 28 (1983) 667–671.

[12] B. Fedosov, Formal quantization, in: *Some Topics of Modern Mathematics and Their Applications to Problems of Mathematical Physics* (Moskow, 1985) pp. 129–136.

[13] B. Fedosov, Index theorems, Itogi Nauki i Tekhuiki. 65 (1991).

[14] B. Fedosov, A simple geometrical constuction of deformation quantization, *J. Differential Geom.* 40 (1994) 213–238.

[15] M. Flato, A. Lichnerowicz and D. Sternheimer, Déformations 1-différentiables d'algèbres de Lie attachées à une variété symplectique ou de contact, Compositio. Math. 31 (1975) 47–92.

[16] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. 78 (1963) 267–288.

[17] M. Gerstenhaber, Deformation theory of algebraic structures, Ann. of Math. 79 (1964) 59–90.

[18] M. Gerstenhaber and S.D. Schack, Quantum symmetry, in: *Quantum Groups*, Lecture Notes in Mathematics, Vol. 1510 (Springer, Berlin, 1992).

[19] A. Lichnerowicz, Déformations d'algèbres associées à une variété symplectique (Les $*_\nu$-Produits), Ann. Inst. Fourier 32 (1982) 157–209.

[20] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, J. Differential Geom. 18 (1983) 523.

[21] C. Moreno, Geodesic symmetries and invariant star products, Lett. Math. Phys. 13 (1987) 245–257.

[22] C. Moreno, Produits star et groupes quantiques. *Notes du cours dans l'Ecole d'Eté du C.I.M.P.A., Quantification – Quantification par déformation* (Nice, France, 1996).

[23] C. Moreno and L. Valero, Produits star invariants et équation de Yang–Baxter quantique constante, *Dans les actes des Journées Relativistes* (Aussois, France, 1990).

[24] C. Moreno, and L. Valero, Star products and quantum groups, in: *Physics on Manifolds, proc. internat. Colloq. in Honour of Yvonne Choquet-Bruhat*, Paris (Kluwer Academic Publishers, Dordrecht, 1992).

[25] R. Nest and B. Tsygan, Algebraic index theorem, Comm. Math. Phys. 172 (1995) 223–262.
[26] R. Nest and B. Tsygan, Algebraic index theorem for families, Adv. in Math. 1133 (1995) 151–205.
[27] M. Rieffel, *Deformation Quantization for actions of* $\mathbb{R}^d$, *Memoirs AMS*, 506 (American Mathematical Society, Providence, RI, 1993).
[28] M. Rieffel, Non-compact quantum groups associated with abelian subgroups, *Comm. Math. Phys.* 171 (1995) 181–201.
[29] L.A. Takhtajan, Lectures on quantum groups, in: *Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory*, Nankai Lectures on Mathematical Physics (World Scientific, London, 1990).
[30] F. Treves, *Topological Vector Spaces, Distributions and Kernels* (Academic Press, New York, 1967).